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A NOTE ON THE G-SPACE VERSION OF GLICKSBERG'S THEOREM

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In an earlier paper, the author generalized Glicksberg's theorem about the Stone-Čech compactifiction of products to the context of G-spaces and their maximal G-compactifications, where G is an arbitrary locally compact group, acting on all spaces under consideration. However, in that paper only products of finitely many factors were considered. In the present note, infinite products are taken into account.

A note on the G-space version of Glicksberg's theorem. This note is a supplement to [2] and [3]. In [2] the theorem below was proved for finite products (with "G-pseudocompact" instead of "pseudocompact"). Later in [3] it was shown that G-pseudocompactness is equivalent to pseudocompactness. Using the result from [3], we are now able to prove the theorem in its full generality. For notation and terminology we refer to [2]. In particular, G is a locally compact topological group and all G-spaces have completely regular Hausdorff phase spaces.

THEOREM. *Let $\{\langle X_\lambda, \pi_\lambda\rangle : \lambda \in \Lambda\}$ be a set of G-spaces. Then the following statements hold true*:

(i) *Suppose G is locally connected and there exists a partition $\Lambda = \Lambda \cup \Delta$ such that both $\prod_{\gamma \in \Gamma} X_\gamma$ and $\prod_{\delta \in \Delta} X_\delta$ are G-infinite. If $\beta_G(\prod_{\lambda \in \Lambda} X_\lambda) = \prod_{\lambda \in \Lambda} \beta_G X_\lambda$, then $\prod_{\lambda \in \Lambda} X_\lambda$ is pseudocompact.*

(ii) *If $\prod_{\lambda \in \Lambda} X_\lambda$ is pseudocompact, then $\beta_G(\prod_{\lambda \in \Lambda} X_\lambda) = \prod_{\lambda \in \Lambda} \beta_G X_\lambda$.*

Proof. (i) In [2] this statement was proved for a product of two factors (note, that by [3] the conclusion of [2] that the product is G-pseudocompact, implies that the product is pseudocompact). So we have to reduce the case of infinite products to the case of a product of two factors. This can be done exactly as in [1], once the following claim has been proved:

Claim. If $\beta_G(\prod_{\lambda \in \Lambda} X_\lambda) = \prod_{\lambda \in \Lambda} \beta_G X_\lambda$, then for every subset Γ of Λ one has $\beta_G(\prod_{\gamma \in \Gamma} X_\gamma) = \prod_{\gamma \in \Gamma} \beta_G X_\gamma$. (This claim holds true without the additional conditions, mentioned in the theorem above.) The proof of this claim cannot be given similar as in (one of the footnotes of) [1], because in general the embedding of a subproduct in the full product cannot be performed in an equivariant way. Instead, we shall use the projection p_Γ: $\prod_{\lambda \in \Lambda} X_\lambda \to \prod_{\gamma \in \Gamma} X_\gamma$, which is equivariant.

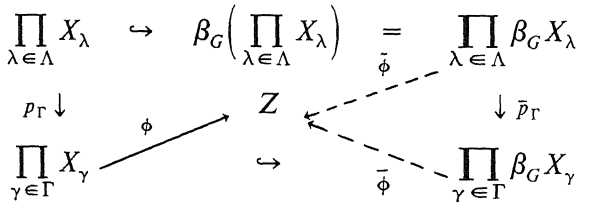
Proof of the claim. We need the following notational convention. If ψ: $Y_1 \to Y_2$ is a continuous mapping between two topological spaces, then $R_\psi := \{(y, y') \in Y_1 \times Y_1 \colon \psi(y) = \psi(y')\}$. If Y_2 is a Hausdorff space, then R_ψ is closed in $Y_1 \times Y_1$. We shall prove, that $\prod_{\gamma \in \Gamma} \beta_G X_\gamma$ has the universal property, which characterizes $\beta_G(\prod_{\gamma \in \Gamma} X_\gamma)$. So let $\langle Z, \zeta \rangle$ be an arbitrary compact Hausdorff G-space and let ϕ: $\prod_{\gamma \in \Gamma} X_\gamma \to Z$ be an equivariant continuous mapping. By assumption, $\phi \circ p_\Gamma$: $\prod_{\lambda \in \Lambda} X_\lambda \to Z$ has a continuous extension $\tilde{\phi}$: $\prod_{\lambda \in \Lambda} \beta_G X_\lambda \to Z$. Let $\bar{p}_\Gamma$ denote the canonical projection of $\prod_{\lambda \in \Lambda} \beta_G X_\lambda$ onto $\prod_{\gamma \in \Gamma} \beta_G X_\gamma$. We want to show that $\tilde{\phi}$ factorizes over $\bar{p}_\Gamma$. To do so, consider the set $R_{\tilde{\phi}} \subseteq \prod_{\lambda \in \Lambda} \beta_G X_\lambda \times \prod_{\lambda \in \Lambda} \beta_G X_\lambda$. By the definition of $\tilde{\phi}$, it is clear that

$$R_{\bar{p}_\Gamma} \cap \Big(\prod_{\lambda \in \Lambda} X_\lambda \times \prod_{\lambda \in \Lambda} X_\lambda \Big) = R_{p_\Gamma} \subseteq R_{\phi \circ p_\Gamma} \subseteq R_{\tilde{\phi}}.$$

As for each $\lambda \in \Lambda$, X_λ is dense in $\beta_G X_\lambda$ this implies that

$$R_{\bar{p}_\Gamma} \overset{\star}{\subseteq} \overline{R_{\bar{p}_\Gamma} \cap \Big(\prod_{\lambda \in \Lambda} \times \prod_{\lambda \in \Lambda} X_\lambda \Big)} \subseteq \overline{R}_{\tilde{\phi}} = R_{\tilde{\phi}}.$$

(Note, that in general for a closed set S and a dense set D of a space Y one need not have $S \subseteq \overline{S \cap D}$, but in this special case the inclusion $\overset{\star}{\subseteq}$ is easily seen to be correct: every (basic) nbd of a point of $R_{\bar{p}_\Gamma}$ meets $R_{\bar{p}_\Gamma} \cap (\prod_{\lambda \in \Lambda} X_\lambda \times \prod_{\lambda \in \Lambda} X_\lambda)$.) From this inclusion it follows immediately that there exists a unique mapping $\bar{\phi}$: $\prod_{\gamma \in \Gamma} \beta_G X_\gamma \to Z$ such that $\tilde{\phi} = \bar{\phi} \circ \bar{p}_\Gamma$; cf. the following diagram:



As $\bar{p}_\Gamma$ is a continuous mapping between compact Hausdorff spaces, it is a quotient mapping, hence continuity of $\bar{\phi} \circ \bar{p}_\Gamma (= \tilde{\phi})$ implies continuity of $\bar{\phi}$. Since $\bar{\phi}$ extends ϕ, the restriction of $\bar{\phi}$ to the dense subset $\prod_{\gamma \in \Gamma} X_\gamma$ of $\prod_{\gamma \in \Gamma} \beta_G X_\gamma$ is equivariant, hence by continuity $\bar{\phi}$ is equivariant. This concludes the proof of the claim.

(ii) In [2], Lemma 5.5 it was noticed that if a G-space $\langle Y, \sigma \rangle$ has Y

pseudocompact, then $\beta_G Y = \beta Y$, the ordinary Stone-Čech compactification of Y. Hence (ii) follows immediately from the classical result of Glicksberg. □

We add some remarks on the "non-triviality condition", mentioned in part (i) of the theorem, i.e. the condition

(C) there is a partition $\Lambda = \Gamma \cup \Delta$ such that $\prod_{\gamma \in \Gamma} X_\gamma$ and $\prod_{\delta \in \Delta} X_\delta$ are G-infinite.

In the case of the classical Glickberg theorem (i.e. G the trivial group) this condition is easily seen to be equivalent to the following one:

(C′) $$\forall \lambda_0 \in \Lambda: \prod_{\lambda \neq \lambda_0} X_\lambda \text{ is } G\text{-infinite.}$$

In the general case one still has (C) ⇒ (C′). Indeed, suppose (C) holds, and that $\lambda_0 \in \Delta$. Then $\Gamma \subseteq \Lambda \setminus \{\lambda_0\}$, so the projection p_Γ: $\prod_{\lambda \neq \lambda_0} X_\lambda \to \prod_{\gamma \in \Gamma} X_\gamma$ is a continuous, equivariant surjection. Taking the preimage under p_Γ of an infinite G-dispersion in $\prod_{\gamma \in \Gamma} X_\gamma$ we see that $\prod_{\lambda \neq \lambda_0} X_\lambda$ is G-infinite. A similar proof deals with the case that $\lambda_0 \in \Gamma$. The converse implication fails in general:

EXAMPLE. Let $X_1 = X_2 = X_3 := \mathbf{R}/\mathbf{Z}$ (the circle) with an action of $\mathbf{R}$ defined by $tx := x + t \pmod 1$ for $t \in \mathbf{R}$, $x \in \mathbf{R}/\mathbf{Z}$. For $n \geq 4$, let X_n be a one-point space with trivial action of $\mathbf{R}$. None of the spaces X_n ($n \in \mathbf{N}$) is $\mathbf{R}$-infinite; in particular, X_1, X_2 and X_3 are not (see [**2**], 2.2(3°)). However, $X_1 \times X_2$ is $\mathbf{R}$-infinite, as are $X_1 \times X_3$ and $X_2 \times X_3$; the idea of proof is similar to the example in the proof of 2.5(iv) ⇒ (iii) in [**2**]. So the family $\{X_n\}_{n \in \mathbf{N}}$ satisfies (C′) but it does not satisfy (C).

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